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A TOPOLOGICALLY STRONGLY MIXING SYMBOLIC MINIMAL SET

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Abstract. Recent papers by the author, Keynes and Robertson, and others have shown that weakly mixing minimal flows are objects of considerable interest, but examples of such flows, other than the horocycle flows, have been scarce. We give here a "machinal" construction of a bilateral sequence with entries from $\{0, 1\}$ whose orbit closure is topologically strongly mixing and minimal. We prove in addition that the flow we obtain has entropy zero, is uniquely ergodic, and fails to be measure-theoretically strongly mixing.

1. **Mixing for symbolic flows.** In this section we give some definitions and establish criteria for the weak and strong mixing of symbolic flows. For unexplained notation and terminology the reader may consult [2], [3], [4], and [7].

Let X be a compact metric space and $\phi: X \to X$ a homeomorphism; the pair (X, ϕ) is called a *flow*. If (X, ϕ) is a flow and $x \in X$, the *orbit* of x is $\mathcal{O}(x) = \{\phi^n x : n \in Z\}$, where Z denotes the set of integers. A flow (X, ϕ) is said to be *minimal* if for each $x \in X$ we have $\mathcal{O}(x)^- = X$, *ergodic* if there is $x \in X$ such that $\mathcal{O}(x)^- = X$, *weakly mixing* if the product flow $(X \times X, \phi)$ is ergodic, and *strongly mixing* if given nonempty open subsets A and B of X there is $n_0 \in Z$ such that $\phi^n A \cap B \neq \varnothing$ whenever $|n| \geqq n_0$. Observe that a flow (X, ϕ) is ergodic if and only if given nonempty open subsets A and B of X there is $n \in Z$ such that $T^n A \cap B \neq \varnothing$. It can be proved [12] that (X, ϕ) is weakly mixing if and only if given nonempty open subsets A and B of X there is $n \in Z$ such that $\phi^n A \cap A \neq \varnothing$ and $\phi^n A \cap B \neq \varnothing$.

Let $\{0, 1\}$ have the discrete topology and let

$$S = \prod_{-\infty}^{\infty} \{0, 1\} = \{0, 1\}^Z,$$

so that S is a compact metric space. We will think of elements $x \in S$ as being bilateral sequences of 0's and 1's, $x = \cdots x_{-2} x_{-1} \dot{x}_0 x_1 x_2 \ldots$, with the dot marking

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the "central position" of the sequence x. A metric compatible with the topology of S is given by, for example, $d(x, y)=1/(1+k)$, where

$$k = \inf\{|n| : x_n \neq y_n\}, \qquad x, y \in S.$$

If k is a positive integer, a *k-block* B will be an ordered k-tuple $B=b_1b_2\cdots b_k$, where each b_i is 0 or 1 for $i=1, 2, \ldots, k$. We will say that C is a *block* in case C is a k-block for some k. If $x \in S$, $B=b_1b_2\cdots b_k$ is a k-block, and $n \in Z$ is such that $x_n=b_1$, $x_{n+1}=b_2, \ldots, x_{n+k-1}=b_k$, then we will say that *B appears at the nth place in x*. The phrase *B appears in x* will mean that there is $n \in Z$ such that B appears at the nth place in x. If k is a nonnegative integer and A is a $(2k+1)$-block which appears at the $(-k)$th place in a sequence $x \in S$, then A will be called *the central $(2k+1)$-block of x*. A block D will be called a *central block* of a sequence $x \in S$ if there is a k such that D is the central $(2k+1)$-block of x. It is not difficult to see that if $x \in S$ then the family $\{\{y \in S\colon A \text{ is a central block of } y\}\colon A \text{ is a central block of } x\}$ forms a neighborhood base at x.

The *shift transformation* $\sigma\colon S \to S$ is defined by $(\sigma x)_n = x_{n+1}$ for all $x \in S$. The map σ is a homeomorphism of S onto S, and thus the pair (S, σ) is a flow. If R is a closed subset of S such that $\sigma R=R$, then the pair $(R, \sigma|R)$, usually written just (R, σ), is called a *symbolic flow*.

If B is a k-block, the *length* of B will be defined to be $L(B)=k$. Let q and p be positive integers with $q \leqq p$, let $E=e_1e_2\cdots e_q$ be a q-block, $F=f_1f_2\cdots f_p$ a p-block, and let $j \in Z$ be such that $1 \leqq j \leqq p-q+1$. We will say that *E appears at the jth place in F* if $f_j=e_1, f_{j+1}=e_2, \ldots, f_{j+q-1}=e_q$. If E and F are blocks, we will say that *E appears in F* in case there is a j such that E appears at the jth place in F.

Let $x \in S$ and let $X=\mathcal{O}(x)^-$. For each block C which appears in x, let $\mathcal{N}(C) = \{n\colon C \text{ appears at the } n\text{th place in } x\}$. If C and D are blocks which appear in x, we define $\mathcal{N}(C, D)=\{m-n : m \in \mathcal{N}(C), n \in \mathcal{N}(D)\}$. The following theorems are stated without proof, since they are little more than direct translations of the definitions of weak and strong mixing into the framework of symbolic dynamics.

THEOREM 1.1. *Let $x \in S$ and let $X=\mathcal{O}(x)^-$. Then the following statements are equivalent*:

(1) *(X, σ) is weakly mixing.*

(2) *For any two blocks A and B which appear in x we have*

$$\mathcal{N}(A, A) \cap \mathcal{N}(B, A) \neq \varnothing.$$

(3) *For each central block A of x and each $L(A)$-block B which appears in x we have*

$$\mathcal{N}(A, A) \cap \mathcal{N}(B, A) \neq \varnothing.$$

THEOREM 1.2. *Let $x \in S$ and let $X=\mathcal{O}(x)^-$. Then the following statements are equivalent:*

(1) *(X, σ) is strongly mixing.*

(2) *For any two blocks A and B which appear in x there is an integer n_0 such that*

$$\mathcal{N}(A, B) \supseteq \{n \in Z : |n| \geqq n_0\}.$$

(3) *For each central block A of x there is an integer n_0 such that*

$$\mathcal{N}(A, A) \supseteq \{n \in Z : |n| \geqq n_0\}.$$

(4) *There is a sequence $\{A_k : k=0, 1, 2, \ldots\}$ of central blocks of x such that $L(A_k) \to \infty$ and such that for each $k=0, 1, 2, \ldots$ there is an integer $n_0(k)$ such that*

$$\mathcal{N}(A_k, A_k) \supseteq \{n \in Z : |n| \geqq n_0(k)\}.$$

2. **Construction of** (W, σ). For any block $A=a_1a_2\cdots a_p$ and for $1 \leqq i \leqq p$ we define $A[i]=a_ia_{i+1}\cdots a_pa_1a_2\cdots a_{p-1}$, the cyclic permutation of A which has its initial point at the ith place of A.

We construct a sequence of blocks A_k in the following manner. Given A_i for $i \leqq k$, A_{k+1} is obtained from A_k by writing down in succession certain "allowable" cyclic permutations $A_k[i]$ of A_k and alternating these with repetitions of A_k itself. The "allowable" cyclic permutations of A_k are those which split no appearance (which is explicit in the construction) of A_i in A_k, for $i<k$. Forbidding certain cyclic permutations of A_k allows us to be certain that the sequence so defined is almost periodic; and there are still enough "allowable" cyclic permutations of A_k available to assure us that the sequence has a weakly mixing orbit closure.

In the following the A_k will be a sequence of central blocks of the element $w \in S$ to be defined, and $\mathfrak{A}(k)$ will denote the set of places in A_k (counted from the left endpoint of A_k) which are allowable as initial points of cyclic permutations of A_k. The B_k will be an auxiliary sequence of blocks used to determine the $\mathfrak{A}(k)$.

Specifically, then, define $A_0=101$, $\mathfrak{A}(0)=\{1, 2, 3\}$, and $B_0=111$. For each k, m_k will denote the cardinal of the set $\mathfrak{A}(k)$, L_k will denote $L(A_k)=L(B_k)$, and E_k will denote the L_k-block all of whose entries are 0 except the first, which is 1.

Assume now that A_k, B_k, and $\mathfrak{A}(k)=\{i_1, i_2, \ldots, i_{m_k}\}$ with $1=i_1<i_2<\cdots<i_{m_k}$ have been defined. Then we define

$$A_{k+1} = A_kA_k[i_{m_k}]A_k\cdots A_kA_k[i_2]A_kA_k[i_2]A_k\cdots A_kA_k[i_{m_k}]A_k$$

and

$$B_{k+1} = E_kB_k[i_{m_k}]E_k\cdots E_kB_k[i_2]E_kB_k[i_2]E_k\cdots E_kB_k[i_{m_k}]E_k.$$

If $B_{k+1}=b_1b_2\cdots b_{L_{k+1}}$, we then define $\mathfrak{A}(k+1)=\{j : 1 \leqq j \leqq L_{k+1} \text{ and } b_j=1\}$.

Thus B_{k+1} has the function of marking the places in A_{k+1} which are allowable as initial points of cyclic permutations of A_{k+1}: if 1 appears at the jth place in B_{k+1}, then $j \in \mathfrak{A}(k+1)$ and $A_{k+1}[j]$ appears in A_{k+2}. We notice also that if $1 \leqq j \leqq L_{k+1}$ and $j \equiv 1 \pmod{L_k}$, then $j \in \mathfrak{A}(k+1)$.

By way of illustration,

$$\begin{aligned} A_1 &= 101\ 110\ 101\ 011\ 101\ 011\ 101\ 110\ 101, \\ B_1 &= 100\ 111\ 100\ 111\ 100\ 111\ 100\ 111\ 100, \text{ and} \\ \mathfrak{A}(1) &= \{1, 4, 5, 6, 7, 10, 11, 12, 13, 16, 17, 18, 19, 22, 23, 24, 25\}. \end{aligned}$$

In this way, then, A_k, B_k, and $\mathfrak{A}(k)$ are defined by induction for all integers $k \geqq 0$. For each $k \geqq 0$ let w_k be that element of S whose central L_k-block is A_k and all of whose other entries are zero. Since $L_k \to \infty$ and the central L_k-subblock of A_{k+1} is A_k, $\lim_k w_k = w$ exists. We put $W = \mathcal{O}(w)^-$.

From the definition of B_{k+1} we observe that $m_{k+1} = 2m_k^2 - 1$ and $L_{k+1} = (4m_k - 3)L_k$. Each m_n is in fact a (2^{n+1})th solution of the Pell equation $x^2 - 2y^2 = 1$. For $(1-\sqrt{2})(1+\sqrt{2}) = -1$, so for $n \geqq 1$ we have $(1-\sqrt{2})^{2^n}(1+\sqrt{2})^{2^n} = 1$. For each $n \geqq 1$ write $(1-\sqrt{2})^{2^n} = x_n - \sqrt{2}y_n$, with x_n, $y_n \in Z$. Then $(1+\sqrt{2})^{2^n} = x_n + \sqrt{2}y_n$, so $1 = (x_n - \sqrt{2}y_n)(x_n + \sqrt{2}y_n) = x_n^2 - 2y_n^2$. However,

$$(1-\sqrt{2})^{2^n} = (x_{n-1} - \sqrt{2}y_{n-1})^2,$$

so we have the relation $x_n = x_{n-1}^2 + 2y_{n-1}^2 = 2x_{n-1}^2 - 1$. Since $x_1 = 3 = m_0$, we may conclude that $m_n = x_{n+1}$ for all integers $n \geqq 0$.

3. **Strong mixing, minimality, and topological entropy.** We prove in this section that the flow (W, σ) is strongly mixing and minimal and has topological entropy zero. We need to know first that our definition of the sequence w employs "enough" cyclic permutations at each stage of the construction.

For each $k = 0, 1, 2, \ldots$ let $\mathfrak{B}(k) = \{j : 1 \leqq j \leqq L_k$ and there is $i \in \mathfrak{A}(k) - \{1\}$ such that 1 appears at the jth place in $B_k[i]\}$. Then if $r \in \mathfrak{B}(k)$, we may split A_{k+1} at the rth place of $A_k[i]$, for some $i \in \mathfrak{A}(k)$, in order to form a permutation $A_{k+1}[j]$ which appears in A_{k+2}.

LEMMA 3.1. *For each* $k = 0, 1, 2, \ldots$, $\mathfrak{B}(k) \supseteq \{n \in Z : 1 \leqq n \leqq L_k\}$.

Proof. We proceed by induction. For $k=0$, $B_0 = B_0[2] = B_0[3] = 111$, so clearly $\mathfrak{B}(0) = \{1, 2, 3\} \supseteq \{n \in Z : 1 \leqq n \leqq L_0\}$. Suppose now that $k \geqq 0$ and

$$\mathfrak{B}(k) \supseteq \{n \in Z : 1 \leqq n \leqq L_k\},$$

and let $n \in Z$ with $1 \leqq n \leqq L_{k+1}$ be given. We need to find $i \in \mathfrak{A}(k+1) - \{1\}$ such that 1 appears at the nth place in $B_{k+1}[i]$.

Let $n' \equiv n \pmod{L_k}$ and $1 \leqq n' \leqq L_k$; suppose $n = n' + pL_k$. By induction there is $j \in \mathfrak{A}(k) - \{1\}$ such that 1 appears at the (n')th place in $B_k[j]$. Now $B_k[j]$ appears in B_{k+1}, since $j \neq 1$. Because $r \equiv 1 \pmod{L_k}$ implies $r \in \mathfrak{A}(k+1)$, and because $B_k[j]$ appears at least twice in B_{k+1}, there is $i \in \mathfrak{A}(k+1) - \{1\}$ such that $B_k[j]$ appears at the $(pL_k + 1)$th place in $B_{k+1}[i]$. But then we see that 1 appears at the nth place in $B_{k+1}[i]$.

THEOREM 3.1. (W, σ) *is strongly mixing.*

Proof. Using Theorem 1.2, it suffices to prove that for each $k = 0, 1, 2, \ldots$ there is an integer $n_0(k)$ such that $\mathcal{N}(A_k, A_k) \supseteq \{n \in Z : |n| \geqq n_0(k)\}$. (Here the A_k's are as in §2.)

Given $k = 0, 1, 2, \ldots$, let $n_0(k) = L_k$ and let $N \in Z$ with $|N| \geqq n_0(k)$ be given. Now the block A_kA_k appears in A_{k+2}, so if $|N| = n_0(k)$, then $N \in \mathcal{N}(A_k, A_k)$;

thus we may assume that $|N| \geqq n_0(k)+1$. Choose m such that $L_m \geqq |N|$, so $L_m > L_k$ and A_k appears in A_m. If $r = L_m - |N| + L_k + 1$, then $1 \leqq r \leqq L_m$, so by the lemma there is $j \in \mathfrak{A}(m) - \{1\}$ such that 1 appears at the rth place in $B_m[j]$. If $A_m[j] = q_1 q_2 \cdots q_{L_m}$, then there is $i \in \mathfrak{A}(m+1)$ such that

$$A_{m+1}[i] = q_r q_{r+1} \cdots q_{L_m} A_m \cdots A_m q_1 q_2 \cdots q_{r-1}.$$

But then the block $A_{m+1}A_{m+1}[i]$ appears in w and contains the block $A_k q_r q_{r+1} \cdots q_{L_m} A_k$, so we see that $L_k + L_m - r + 1 \in \mathcal{N}(A_k, A_k)$. Since $L_k + L_m - r + 1 = |N|$, we have $N \in \mathcal{N}(A_k, A_k)$.

In order to prove that the sequence w is almost periodic, we need to know that the appearances of A_i which are explicit in the construction are not split as we proceed with the construction of w. To this end we introduce another auxiliary sequence of blocks Λ_k, with entries from Z. Define $\Lambda_0 = (-1)(-1)(-1)$ and $\Lambda_0' = (0)(-1)(-1)$, where the parentheses serve only to separate the entries in a block from one another. If k is a nonnegative integer and Λ_k and Λ_k' have been defined, define

$$\Lambda_{k+1} = \Lambda_k' \Lambda_k[i_{m_k}] \Lambda_k' \cdots \Lambda_k' \Lambda_k[i_2] \Lambda_k' \Lambda_k[i_2] \Lambda_k' \cdots \Lambda_k' \Lambda_k[i_{m_k}] \Lambda_k',$$

where the i_j's are as in §2. If $\Lambda_{k+1} = \lambda_1 \lambda_2 \cdots \lambda_s$ with each $\lambda_i \in Z$, define $\Lambda_{k+1}' = (k+1)\lambda_2\lambda_3 \cdots \lambda_s$. Thus Λ_k and Λ_k' are defined by induction for all integers $k \geqq 0$. Note that $L(\Lambda_k) = L(\Lambda_k') = L_k$ for all k.

Let m and k be integers with $m \geqq k \geqq 0$ and suppose that $\Lambda_m' = \lambda_1 \lambda_2 \cdots \lambda_s$ with each $\lambda_i \in Z$. The following statements are established by straightforward induction arguments.

(1) If $\lambda_j \geqq k$, then the L_k-subblock of B_m appearing at the jth place in B_m is either $000 \cdots 00$ or $100 \cdots 00$.

(2) If $\lambda_j \geqq k$, then the L_k-subblock of A_m appearing at the jth place in A_m is A_k.

(3) If $1 \leqq i \leqq j \leqq L_m$, $\lambda_i \geqq k$ and $\lambda_j \geqq k$, and if $i \leqq p \leqq j$ implies $\lambda_p < k$, then $j - i \leqq L_k$.

These three statements imply that we may write $A_m = A_k C_1 A_k C_2 A_k \cdots A_k C_r A_k$, where $L(C_j) \leqq L_k$ for $j = 1, 2, \ldots, r$. Therefore A_k appears in each A_m, and hence in w, with bounded gap. It follows that the sequence w is almost periodic, and so we may state the following theorem.

THEOREM 3.2. *(W, σ) is minimal.*

For each $n = 1, 2, \ldots$ let $\theta(n)$ denote the number of different n-blocks which appear in the sequence w. Then the limit

$$K(W) = \lim_{n \to \infty} \frac{\log \theta(n)}{n}$$

exists and is the *topological entropy* (see [1], [2], and [4]) of the flow (W, σ). We wish to prove that $K(W) = 0$. For this purpose it suffices to prove that

$$\lim_{k \to \infty} \frac{\log \theta(L_k)}{L_k} = 0.$$

LEMMA 3.2. *If k is a nonnegative integer and B is an L_k-block which appears in w, then B appears in A_{k+2}.*

Proof. Let $m>k$. Then from the discussion preceding Theorem 4.2 we know that we may write

$$A_m = A_k C_1 A_k C_2 A_k \cdots A_k C_r A_k,$$

where $L(C_i) \leqq L_k$ for $i=1, 2, \ldots, r$. A straightforward proof by induction yields that each C_i has the following property (*): C_i is a subblock of $A_k[j]$ for some $j \in \mathfrak{A}(k)$ such that if $B_k[j]=b_1 b_2 \cdots b_{L_k}$, $A_k[j]=a_1 a_2 \cdots a_{L_k}$, and $C_i = a_p a_{p+1} \cdots a_{p+s}$, then $b_p = b_{p+s+1} = 1$, where $p+s+1$ is to be reduced modulo L_k in case $p+s+1 > L_k$. It is easy to see that A_{k+2} contains all blocks of the form $A_k D A_k$, where D is a block having property (*). Therefore each block $A_k C_i A_k$ appears in A_{k+2}. Now if B is an L_k-block which appears in w, then B appears in A_m for some $m>k$. Writing A_m as above, we see that B must appear in one of the blocks $A_k C_i A_k$. Since each block $A_k C_i A_k$ appears in A_{k+2}, B must appear in A_{k+2}.

THEOREM 3.3. *The topological entropy of (W, σ) is zero.*

Proof. Lemma 3.2 implies that $\theta(L_k) \leqq L_{k+2}$. Recalling that $m_{k+1} = 2m_k^2 - 1$ and $L_{k+1} = (4m_k - 3)L_k$, we see that

$$\theta(L_k) \leqq L_{k+2} = (4(2m_k^2-1)-3)(4m_k-3)L_k \leqq 32 m_k^3 L_k,$$

so

$$\frac{\log \theta(L_k)}{L_k} \leqq \frac{\log 32}{L_k} + \frac{3 \log m_k}{L_k} + \frac{\log L_k}{L_k}.$$

Since $L_k \to \infty$, it suffices to prove that

$$\lim_{k \to \infty} \frac{\log m_k}{L_k} = 0.$$

But

$$\begin{aligned} \frac{\log m_k}{L_k} &= \frac{\log(2m_{k-1}^2 - 1)}{(4m_{k-1}-3)L_{k-1}} \leqq \frac{\log(m_{k-1}^3)}{(4m_{k-1}-3)L_{k-1}} \\ &= \frac{3}{4m_{k-1}-3} \frac{\log m_{k-1}}{L_{k-1}} \leqq \cdots \leqq \frac{3^k}{\prod_{i=0}^{k-1}(4m_i - 3)} \frac{\log m_0}{L_0} \\ &\leqq \frac{3^k}{\prod_{i=0}^{k-1}(4m_0-3)} \frac{\log m_0}{L_0} = \frac{3^k}{9^k} \frac{\log 3}{3} \to 0. \end{aligned}$$

4. **Unique ergodicity and failure of measure-theoretic strong mixing.** We wish to prove in this section that the flow (W, σ) admits a unique normalized invariant Borel measure μ, and that the process [2] $(W, \mathscr{B}, \mu, \sigma)$, where $\mathscr{B}$ is the collection of all Borel subsets of W, is not strongly mixing.

In order to prove that (W, σ) is uniquely ergodic, it suffices to prove that the defining sequence $w \in W$ is *strictly transitive* [10]. Specifically, for any two blocks B and C which appear in w, let $\nu(B, C)$ denote the number of times that B appears in C; i.e., $\nu(B, C) = \text{card}\,\{j\colon B$ appears at the jth place in $C\}$. Then w is strictly transitive if and only if given a block B which appears in w and $\varepsilon > 0$, there is a k such that if C is any L_k-block which appears in w then

$$|\nu(B, C) - \nu(B, A_k)|/L_k < \varepsilon.$$

THEOREM 4.1. *(W, σ) is uniquely ergodic.*

Proof. Let a block B which appears in w, say with $L(B) = p$, and $\varepsilon > 0$ be given. Since $L_{k-1}/L_k \to 0$ and $m_{k-1} \leqq L_{k-1}$, we may choose k such that B appears in A_{k-1}, $8(L_{k-1}/L_k) < \varepsilon/3$, $8p(m_{k-1}/L_k) < \varepsilon/3$, and $6(p/L_k) < \varepsilon/3$. Let C be any L_k-block which appears in w. We wish to prove that

$$|\nu(B, C) - \nu(B, A_k)|/L_k < \varepsilon. \tag{1}$$

To this end we first need to estimate $\nu(B, A_k)$ in terms of $\nu(B, A_{k-1})$.

Taking note of the definition of A_k in terms of A_{k-1}, of the fact that

$$|\nu(B, A_{k-1}[j]) - \nu(B, A_{k-1})| \leqq p$$

for all $j \in \mathfrak{A}(k-1)$, and of the fact that there are $4(m_{k-1}-1)$ "junctures" of the form $A_{k-1}A_{k-1}[j]$ or $A_{k-1}[j]A_{k-1}$ in the definition of A_k (and that the block B can appear no fewer than 0 and no more than p times across any one of these junctures), we conclude that there are $r \in [-1, 1]$ and $s \in [0, 1]$ such that

$$\nu(B, A_k) = (4m_{k-1}-3)\nu(B, A_{k-1}) + 2p(m_{k-1}-1)(r+2s). \tag{2}$$

Now C is an L_k-block which appears in w and hence, by Lemma 3.2, C appears in A_{k+2}. There are now five cases to consider, depending on just which subblock of A_{k+2} the block C happens to be. If $C = A_k$ or $C = A_k[i]$ for some $i \in \mathfrak{A}(k)$, then the verification of (1) is immediate. Suppose then that C appears in a block $A_kA_k[i]$ for some $i \in \mathfrak{A}(k)$; the argument in case C appears in a block $A_k[i]A_k$ will be strictly similar and therefore will not be given here. Now

$$A_kA_k[i] = \underbrace{\overbrace{A_{k-1}\cdots A_{k-1}A_{k-1}[j]A_{k-1}}^{A_k}\,\overbrace{DA_{k-1}A_{k-1}[n]A_{k-1}\cdots}^{A_k[i]}}_{C},$$

where D is a subblock of an allowable permutation of A_{k-1}. Counting the minimum and maximum possible numbers of appearances of A_{k-1}, of the $A_{k-1}[j]$'s, and of junctures $A_{k-1}A_{k-1}[j]$ or $A_{k-1}[j]A_{k-1}$ in C yields the estimate

$$\begin{aligned}(2m_{k-1}-3)\nu(B, A_{k-1}) + (2m_{k-1}-4)[\nu(B, A_{k-1}) - p] &\leqq \nu(B, C)\\ &\leqq (2m_{k-1}+1)\nu(B, A_{k-1}) + 2m_{k-1}[\nu(B, A_{k-1})+p] + (4m_{k-1}+1)p.\end{aligned}$$

With the help of (2) this implies that

$$|\nu(B, C) - \nu(B, A_k)| \leqq 3\nu(B, A_{k-1}) + p(8m_{k-1}-1) \leqq 3L_{k-1} + 8pm_{k-1},$$

from which (1) is immediate. The final case to consider is that C appears in a block

A_kDA_k, where D is a subblock of $A_k[r]$ for some $r \in \mathfrak{A}(k)$. A counting procedure similar to the one used in the previous case also establishes (1) for this case. We conclude then that the flow (W, σ) is uniquely ergodic.

Thus there is a unique normalized invariant measure μ on W, and σ is an ergodic measure-preserving transformation of the measure space $(W, \mathscr{B}, \mu)$. In order to show that the process $(W, \mathscr{B}, \mu, \sigma)$ is not strongly mixing, it is sufficient to display two measurable subsets A and B of W for which

$$\lim_{n\to\infty} \mu(\sigma^n A \cap B) \neq \mu(A)\mu(B).$$

To this end we will need some quantitative information about the measure μ.

Let B be a block which appears in the sequence w and let

$$G(B, n) = \{x \in W\colon B \text{ appears at the } n\text{th place in } x\}.$$

Since $\sigma G(B, n) = G(B, n-1)$ and σ preserves the measure μ, we see that $\mu(G(B, n))$ is independent of n; we therefore denote $\mu(G(B, n))$ by $\mu'(B)$. It is well known that

$$\mu'(B) = \lim_{k\to\infty} \frac{\nu(B, A_k)}{L_k}.$$

For each $k=0, 1, 2, \ldots$ let $Q_k = G(A_k, -\frac{1}{2}(L_k-1))$, and let us estimate

$$\mu(Q_k) = \mu'(A_k) = \lim_{n\to\infty} \frac{\nu(A_k, A_{n+1})}{L_{n+1}}.$$

We have seen earlier that if $m \geqq n$ and $\Lambda' = \lambda_1\lambda_2 \cdots \lambda_s$, then $\lambda_j \geqq k$ implies that the L_k-subblock of A_m appearing at the jth place in A_m is A_k. Let $N(k, m) =$ card $\{j\colon$ the jth entry in Λ'_m is no less than $k\}$. Then $N(k, m) \leqq \nu(A_k, A_m)$ and, from the definition of Λ'_m, $N(k, n+1) = (4m_n-3)N(k, n)$ and $N(k, k+1) = 2m_k - 1$. Thus we have

$$\begin{aligned} \nu(A_k, A_{n+1})/L_{n+1} &\geqq (1/L_{n+1})(4m_n-3)(4m_{n-1}-3)\cdots(4m_{k+1}-3)(2m_k-1) \\ &= (2m_k-1)/L_{k+1}, \end{aligned}$$

and hence $\mu(Q_k) \geqq (2m_k-1)/L_{k+1}$.

THEOREM 4.2. *$(W, \mathscr{B}, \mu, \sigma)$ is not strongly mixing.*

Proof. For each $k=0, 1, 2, \ldots$ and $n=0, 1, 2, \ldots$ let

$$Q_{k,n} = (\sigma^{2L_n}Q_k) \cap Q_k.$$

We will show that there is a k for which

$$\liminf_{n\to\infty} \mu(Q_{k,n}) > (\mu(Q_k))^2,$$

and this will show that $(W, \mathscr{B}, \mu, \sigma)$ cannot be strongly mixing.

For each m, n, and k with $m > n > k$ let $\nu(m, n, k)$ denote the number of blocks of the form A_kDA_k, where $L(D)=2L_n-L_k$, which appear in A_m. Then it is easy to see that

$$\mu(Q_{k,n}) = \lim_{m\to\infty} \frac{\nu(m, n, k)}{L_m}.$$

Let $p\geqq r$ and $\Lambda_p=\lambda_1\lambda_2\cdots\lambda_s$; we will say that A_r *appears explicitly at the jth place in* A_p if $\lambda_j\geqq r$. Now each time A_{n+1} appears in A_m, A_n appears explicitly $2m_n-1$ times in A_{n+1}. Suppose that A_n appears explicitly at the jth place in A_{n+1} and that $j<L_{n+1}-L_n$; then A_n also appears explicitly at the $(2L_n+j)$th place in A_{n+1}. Therefore, every time that A_k appears in such an appearance of A_n in A_{n+1}, there is another appearance of A_k in A_{n+1} $2L_n$ places along to the right of it. That is to say,

$$\nu(m, n, k) \geqq N(n+1, m)(2m_n-2)\nu(A_k, A_n).$$

It follows then that

$$\nu(m, n, k)/L_m = (2m_{n+1}-1)(2m_n-2)\nu(A_k, A_n)/L_{n+2},$$

and hence

$$\mu(Q_{k,n}) \geqq (2m_{n+1}-1)(2m_n-2)\nu(A_k, A_n)/L_{n+2}.$$

Therefore

$$\begin{aligned}\liminf_{n\to\infty} \mu(Q_{k,n}) &\geqq \liminf_{n\to\infty} \frac{(2m_{n+1}-1)(2m_n-2)\nu(A_k, A_n)}{L_{n+2}}\\ &= \liminf_{n\to\infty} \frac{(2m_{n+1}-1)(2m_n-2)\nu(A_k, A_n)}{(4m_{n+1}-3)(4m_n-3)L_n}\\ &= \frac{1}{4}\lim_{n\to\infty} \frac{\nu(A_k, A_n)}{L_n} = \frac{1}{4}\mu(Q_k).\end{aligned}$$

Since $\bigcap_{k=0}^{\infty} Q_k=\{w\}$ and $Q_{k+1}\subseteq Q_k$, we must have $\lim \mu(Q_k)=0$. Choose Q_k such that $\mu(Q_k)<\frac{1}{4}$. Then evidently

$$\liminf_{n\to\infty} \mu(Q_{k,n}) \geqq \tfrac{1}{4}\mu(Q_k) > (\mu(Q_k))^2,$$

so the process $(W, \mathscr{B}, \mu, \sigma)$ cannot be strongly mixing.

5. **Remarks.** The construction given above can be varied in many ways to give further examples of strongly mixing minimal sets. For example, in defining A_{k+1} the permutations $A_k[j]$ do not have to appear in any particular order, and the A_k's can be repeated more than once between each $A_k[i]$ and $A_k[j]$; the main requirement is that all allowable permutations should appear and forbidden permutations should be excluded. In general, we may choose a sequence $\{m_k : k=0, 1, 2, \ldots\}$ of integers such that $m_k\geqq 1$ for all k and a block A_0 with $L(A_0)=L_0\equiv 1 \pmod 2$, and then for each $n\geqq 0$ define

$$I_n = \{-m_n, -m_n+1, \ldots, -1, 0, 1, \ldots, m_n-1, m_n\},$$

$$L_{n+1} = (2m_n+1)L_n, \quad \text{and} \quad J_n=\{1, 2, \ldots, L_n\}.$$

If for each n we choose a map $f_n\colon I_n \to J_n$ such that $f_n(0)=1$ and define

$$A_{n+1} = A_n[f_n(-m_n)]\cdots A_n[f_n(-1)]A_n[f_n(0)]A_n[f_n(1)]\cdots A_n[f_n(m_n)]$$

for all $n \geqq 0$, then the sequence $\{A_n\}$ of expanding central blocks will define an element $x \in S$, which may be called an *epicyclic sequence*. One-sided sequences can be defined in an analogous manner; in fact, Morse's construction of the Morse-Thue sequence [9] is of this type. As we have seen, many epicyclic sequences will have minimal and strongly mixing orbit closures. Also, homomorphic images of (W, σ) under block mappings [5] will be minimal and strongly mixing.

Finally, we conjecture that the process $(W, \mathscr{B}, \mu, \sigma)$ is measure-theoretically weakly mixing.

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